

# Spin diffusive modes and thermal transport in neutron-star crusts

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## 1 Introduction

In this contribution we review a method for deriving collective modes of pair-correlated neutron matter as found in the inner crust of a neutron star. We discuss two classes of modes lying either above or below the pair-breaking continuum limit  $2\Delta$  and having energy spectra of the form  $\varepsilon = \omega_0 + \alpha q^2$ , where  $\mathbf{q}$  is the wave vector of the mode and  $\omega_0 = 2\Delta$  is a threshold frequency determined by the pairing gap  $\Delta$ . One class of modes, corresponding to density oscillations, arises in the spectrum for small  $q$  when  $\alpha < 0$ . These modes can be associated with undamped particle-hole bound pairs – *excitons* – existing in superfluid neutron matter. We also discuss the modes with  $\alpha > 0$ . These are *spin diffusive modes* that arise in neutron-star crusts due to spin perturbations and are located above the pair-breaking threshold. In contrast to excitons, these modes are damped. As an application we compute the thermal conductivity due to the spin diffusive modes existing in the crust of a neutron star. The general formalism that we introduce below is relevant for a number of fermionic ensembles experiencing attractive interactions, which become superfluid or superconducting below a critical temperature  $T_c$ . Examples are ultra-cold atomic vapors, nuclear and neutron matter, and deconfined quark matter (for review articles see Ref. [1]).

Microscopic description of the low-energy dynamics of superfluid systems can be formulated within the propagator formalism, where the central quantity is the response function. Several types of response functions may be considered, depending on the problem of interest. For neutral fermionic systems, the central quantities are the density and spin responses. For charged superconductors the charge current response has been especially important historically, since it describes the experimentally observed Meissner effect. In the present work we focus instead on the pair-breaking response of a neutral, non-relativistic superfluid exhibiting  $S$ -wave pairing, a response which is dominant at temperatures slightly below the critical temperature of the superfluid phase transition.

Collective modes of neutron-star inner crusts, which are composed of an ionic lattice, a charge-neutralizing electron background, and a superfluid neutron fluid, have been studied from a number of perspectives. The electron-ion system oscillates at the plasma frequency in the absence of the neutron fluid [2]. However, coupling of the lattice phonons to the neutron fluid renormalizes the spectrum of lattice oscillations [3] and may lead to instabilities [4]. The low-lying oscillatory modes with spectra of acoustic type, notably the Anderson-Bogolyubov mode and the lattice phonon mode, tend to mix since they cover the same energy range. Their properties and mixing have been investigated in Refs. [5, 6, 7, 8, 9]. In a first approximation, the pair-breaking modes can be treated separately, as they arise near the pair-breaking threshold  $2\Delta$ . Before discussing concrete applications to neutron-star crusts, we first outline the general formalism for deriving these modes, following closely Ref. [8].

## 2 Formalism

The techniques we employ were first developed by Abrikosov and Gor'kov (AG) to study the Meissner effect in the electrodynamics of dirty superconductors [10]. In this theory the electromagnetic response of the system is expressed in terms of Matsubara imaginary-time propagators, assuming a contact pairing interaction. From the formal standpoint, the AG theory can be viewed as a reformulation of the theories of collective excitations in superconductors developed by Bogolyubov, Anderson, and subsequently others using the second-quantized operator formalism. The ideas of Landau Fermi-liquid theory (which is applicable to fermionic systems with arbitrarily strong interactions) were extended to superfluid systems by Larkin and Migdal and by Leggett [11, 12]. They reproduced the basic results known at the time, going on to derive the so-called Fermi-liquid corrections that arise in strong-coupling theories. In contrast to the AG treatment, the latter approaches distinguish particle-hole (ph) and particle-particle (pp) interactions having different strength/or sign, in the spirit of the Fermi-liquid theory. There is renewed interest in studying the response of superfluid/superconducting nuclear matter, in part with the objective of understanding the collective modes of such systems (see Ref. [8] and works cited therein).

In contributing to this trend, we begin by considering  $S$ -wave paired neutron matter and write the interactions in the particle-particle and particle-hole channel in the forms [11]

$$\hat{f}_{\alpha\beta\gamma\delta}^{\text{pp}} \simeq f_{\text{pp}}^D (i\sigma_2)_{\alpha\beta} (i\sigma_2)_{\gamma\delta} + f_{\text{pp}}^S (i\sigma_2 \boldsymbol{\sigma})_{\alpha\beta} \cdot (\boldsymbol{\sigma} i\sigma_2)_{\gamma\delta}, \quad (1)$$

$$\hat{f}_{\alpha\beta\gamma\delta}^{\text{ph}} \simeq f_{\text{ph}}^D \delta_{\alpha\beta} \delta_{\gamma\delta} + f_{\text{ph}}^S \boldsymbol{\sigma}_{\alpha\beta} \cdot \boldsymbol{\sigma}_{\gamma\delta}. \quad (2)$$

Here  $f^D$  and  $f^S$  are constants that parametrize the two-body interaction in the density and spin channels, respectively, the superscripts pp and ph refer to the particle-

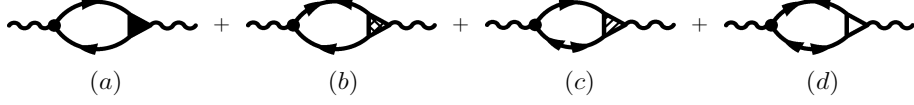


Figure 1: Sum of polarization tensors contributing to the response function of a superfluid. The diagrams  $b$ ,  $c$ , and  $d$  are specific to superfluid systems and vanish in the unpaired state.

particle and particle-hole channels, while the quantities  $\sigma_{\alpha\beta}$  are Pauli matrices in the spin space.

The low-lying excitations of neutron matter can be obtained by expanding the relevant response functions in small parameters of the theory, for example  $q/k_F$ , where  $k_F$  is the Fermi wave number and  $q$  is the magnitude of the momentum transfer. If we are interested in radiation processes, i.e., in the time-like kinematical domain, the quantity  $qv_F/\omega$  is small, where  $v_F$  is the Fermi velocity and  $\omega$  is the energy transfer (typically of the order of the temperature).

As there are four distinct propagators in the theory of superfluid systems (the so-called Nambu-Gor'kov structure), the response function is given by four different contributions, shown as Feynman diagrams in Fig. 1. We adopt the standard notation of the diagram technique for superfluid systems, following Ref. [10]. The shaded (full) vertices appearing in the polarization loops can be computed, in turn, by resummation of infinite loop diagrams, which account for the modifications due to the interactions in the medium (see for example Ref. [8]). To leading order in a small-momentum expansion, the temporal (00) and spatial ( $jj$ ) components of density ( $D$ ) and spin ( $S$ ) response functions may be written as

$$\Pi_D^{00} = -\frac{4q^4 v_F^4}{45\omega^4} \mathcal{G}, \quad \Pi_D^{jj} = -\frac{2q^2 v_F^4}{9\omega^2} \mathcal{G}, \quad (3)$$

$$\Pi_S^{00} = -v_F^2 \mathcal{G}, \quad \Pi_S^{jj} = -\frac{q^2 v_F^2}{\omega^2} \mathcal{G}, \quad (4)$$

where

$$\mathcal{G}(\mathbf{v}, \omega, \mathbf{q}) = \Delta^2 \int_{-\infty}^{+\infty} d\xi_p \left[ \frac{\epsilon_+ - \epsilon_-}{\epsilon_+ \epsilon_-} \frac{f(\epsilon_-) - f(\epsilon_+)}{\omega^2 - (\epsilon_+ - \epsilon_-)^2 + i\eta} - \frac{\epsilon_+ + \epsilon_-}{\epsilon_+ \epsilon_-} \frac{1 - f(\epsilon_-) - f(\epsilon_+)}{\omega^2 - (\epsilon_+ + \epsilon_-)^2 + i\eta} \right].$$

Here  $f(x) = \{\exp[(x - \mu)/T] + 1\}^{-1}$  is the Fermi function,  $\mu$  being the chemical potential. We define  $\epsilon_{\pm} = \sqrt{\xi_{\pm}^2 + \Delta^2}$  and  $\xi_{\pm} = \xi_p \pm \mathbf{q}\mathbf{v}/2$ , where  $\xi_p$  is the spectrum in the unpaired state,  $\Delta$  is the pairing gap,  $\mathbf{v}$  is the particle velocity, and  $\mathbf{q}$  is the momentum transfer. In concentrating on the pair-breaking contribution, we need only keep the term proportional to  $1 - f(\epsilon_-) - f(\epsilon_+)$ .

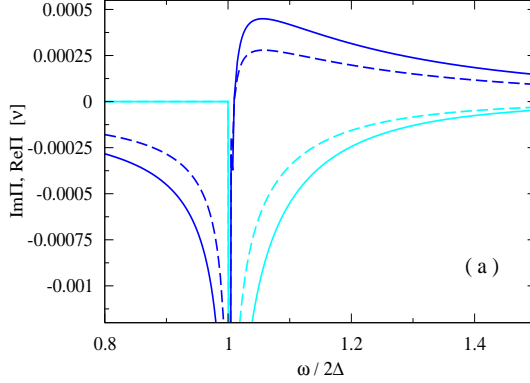


Figure 2: Comparison between numerical (solid lines) and perturbative (dashed lines) results for the density response function  $\Pi_{00}^D$ , in units of the density of states  $\nu$  [8]. The imaginary part of the response function is shown by a heavy (blue) line, the real part by a light (cyan) line. The energy transfer  $\omega$  is in units of  $\omega_0$ . The temperature is  $0.5T_c$  and the pairing gap is  $\Delta = 1.0$  MeV. The ratio of momentum transfer and Fermi momentum is kept fixed at  $q/k_F = 0.01$ , and the Fermi momentum is  $k_F = 1.0 \text{ fm}^{-1}$  (which corresponds to a density  $n = 0.22n_0$  in terms of the nuclear saturation density  $n_0$ .)

Figure 2 illustrates the dependence on the transferred energy  $\omega$ , of the real and imaginary parts of the density response function of neutron matter, for fixed three-momentum transfer. The zero-temperature gap is related to  $T_c$  via  $T_c = \Delta(0)/1.76$ . The Landau parameter is set to  $f_{\text{ph}}^D = -0.5$  for density perturbations, while the frequency and momentum transfer are normalized to the threshold frequency  $\omega_0 = 2\Delta(T)$ . The response function in the negative energy range can be obtained from the relations  $\text{Re}\Pi^D(-\omega) = \text{Re}\Pi^D(\omega)$  and  $\text{Im}\Pi^D(-\omega) = -\text{Im}\Pi^D(\omega)$ . There exists a threshold in the response function because an energy  $\omega \geq \omega_0$  is needed to break a Cooper pair, i.e., the energy transfer must exceed the binding energy of the pair. For energies above the threshold ( $\omega > \omega_0$ ), a non-zero imaginary part implies that the collective excitations have finite life-time. They are not perfect quasiparticles.

### 3 Numerical results and collective modes

The numerical results reported here were obtained in Ref. [8]. Once the response function is determined, we are in a position to write down the associated spectral function, which is defined by  $R(\omega, \mathbf{q}) = -2\text{Im}\Pi(\omega, \mathbf{q})$  and can be cast as

$$R(\omega, \mathbf{q}) = -\frac{2(v_{\text{ph}}^D)^{-2}\text{Im}P(\omega, \mathbf{q})}{[(v_{\text{ph}}^D)^{-1} - \text{Re}P(\omega, \mathbf{q})]^2 + \text{Im}P(\omega, \mathbf{q})^2}, \quad (5)$$

where  $P(\omega, \mathbf{q}) \equiv \mathcal{Q}^+(\omega, \mathbf{q})/\mathcal{C}(\omega, \mathbf{q})$  and

$$\mathcal{Q}^\pm(\omega, \mathbf{q}) = \mathcal{A}^\pm(\omega, \mathbf{q})\mathcal{C}(\omega, \mathbf{q}) - \mathcal{B}(\omega, \mathbf{q})\mathcal{D}^\pm(\omega, \mathbf{q}). \quad (6)$$

The elementary loops are given by [11, 12]

$$\mathcal{A}^\pm = \nu \int \frac{d\Omega}{4\pi} \left\{ -\frac{1 \pm \hat{\mathcal{P}}}{2} \mathcal{G}(\mathbf{v}, \omega, \mathbf{q}) + \frac{\mathbf{qv}}{\omega - \mathbf{qv}} \left[ \mathcal{G}(\mathbf{v}, \mathbf{qv}, \mathbf{q}) - \mathcal{G}(\mathbf{v}, \omega, \mathbf{q}) \right] \right\}, \quad (7)$$

$$\mathcal{B} = -\nu \int \frac{d\Omega}{4\pi} \frac{\omega + \mathbf{qv}}{2\Delta} \mathcal{G}(\mathbf{v}, \omega, \mathbf{q}), \quad (8)$$

$$\mathcal{C} = \nu \int \frac{d\Omega}{4\pi} \frac{\omega^2 - (\mathbf{qv})^2}{4\Delta^2} \mathcal{G}(\mathbf{v}, \omega, \mathbf{q}), \quad (9)$$

$$\mathcal{D}^\pm = \nu \int \frac{d\Omega}{4\pi} \left[ \frac{\omega + \mathbf{qv}}{4\Delta} + \frac{\omega - \mathbf{qv}}{4\Delta} \hat{\mathcal{P}} \right] \mathcal{G}(\mathbf{v}, \omega, \mathbf{q}), \quad (10)$$

where the operator  $\hat{\mathcal{P}}$  is equal to +1 for vertices which are even under the time reversal operation and -1 for vertices odd under this transformation, while  $\nu$  is the density of states.

For small imaginary parts the spectral function can be approximated as

$$R(\omega, \mathbf{q}) = 2\pi Z(\mathbf{q})\delta((f_{\text{ph}}^D)^{-1} - \text{Re}P(\omega, \mathbf{q})) + R_{\text{reg}}(\omega, \mathbf{q}), \quad (11)$$

where  $R_{\text{reg}}(\omega, \mathbf{q})$  is the regular (*i.e.* smooth) part of the spectral function and  $Z(\mathbf{q})$  is the wave-function renormalization [8]. The Dirac delta gives the spectrum of the quasiparticle excitations. In Eq. (11) the interaction term  $f_{\text{ph}}^D$  indicates that one is dealing with density perturbations.

The dispersion relations of the excitations are determined by solutions  $\omega(\mathbf{q})$  of the equation

$$1 - f_{\text{ph}}^D \text{Re}P(\omega, \mathbf{q}) = 0. \quad (12)$$

Fig. 3 demonstrates its solutions in the case of density and spin excitations for the range of interaction values  $-1 \leq f_{\text{ph}}^D \leq 2$ . It turns out that if the interaction has a positive sign the modes are located in the domain  $\omega/\omega_0 > 1$ , where  $\text{Im}P(\omega, \mathbf{q}) \neq 0$ . This is the case for spin excitations in the crust of a neutron star. Such modes are termed *diffusive*. The density modes are located in the domain  $\omega/\omega_0 \leq 1$ , since the particle-hole interaction takes negative values in neutron-star crusts. The pair-breaking part of  $\text{Im}P(\omega, \mathbf{q})$  vanishes in this domain. These modes can be called *excitonic* in that they can be viewed as bound pairs of particles and holes (and are analogous in this respect to modes occurring in ordinary semiconductors). It must be noted that we cannot follow the high-momentum behavior of the modes because we are using perturbative response functions, which are no longer valid at

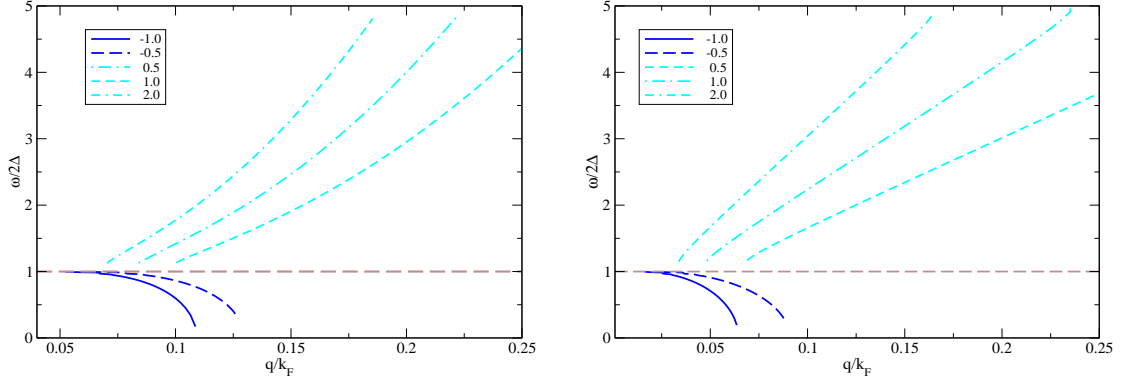


Figure 3: Dependence of the frequency of the modes on momentum transfer for density (left panel) and spin-density (right panel) perturbations [8]. The values of particle-hole interaction  $f_{\text{ph}}$  are shown in each panel. The system parameters are fixed at  $k_F = 1 \text{ fm}^{-1}$ ,  $\Delta = 1.0 \text{ MeV}$ , and  $T/T_c = 0.5$ . Heavy lines (blue online) correspond to undamped excitonic modes. Light lines (cyan online) correspond to diffusive damped modes.

large momentum transfers  $q/k_F \sim 0.3$ . We also observe that for negative  $f_{\text{ph}}$ , the modes tend toward zero with increasing momentum transfer. The high-momentum behavior of the excitonic modes requires further study in order to realize a complete picture of their thermodynamics. In the following section we will concentrate on the properties of the spin diffusive modes.

## 4 Thermal conductivity of spin diffusive modes in neutron-star crusts

We conclude with an estimate of the thermal conductivity of spin diffusive modes, assuming that this property is dominated by the mode-mode scattering process. To treat the thermal transport in the presence of a thermal gradient  $\nabla T$ , we write the kinetic equation for spin diffusive modes as

$$\frac{\partial \varepsilon}{\partial \mathbf{p}} \frac{\partial f_0}{\partial T} \nabla T = I[f], \quad (13)$$

where  $f_0$  is the equilibrium distribution of the modes, which can be taken as a Boltzmann distribution owing to the large value of  $\omega_0$  and  $I[f]$  is the collision integral. Assuming a small perturbation, we linearize the kinetic equation in the standard

fashion, i.e., we substitute

$$f = f_0 + \delta f, \quad \delta f = -\frac{\partial f_0}{\partial \varepsilon} \chi = \frac{f_0}{T} \chi. \quad (14)$$

in the collision integral. For this generic form of the perturbation, the collision integral has the form

$$I[f_1] = \frac{f_{01}}{T} \int w' f_{02} (\chi'_1 + \chi'_2 - \chi_1 - \chi_2) d\Gamma'_1 d\Gamma_2 d\Gamma'_2, \quad (15)$$

where  $d\Gamma_1$  etc. denote the phase space volume measure and  $w'$  is the transition probability. For thermal conductivity the perturbation is of the form

$$\chi = \mathbf{g} \cdot \nabla T = \mathbf{v} \cdot \nabla T \sum_{s=1}^{\infty} A_s S_{3/2}^s(\beta \mathbf{v}), \quad (16)$$

where  $\mathbf{v} = (\partial \varepsilon / \partial \mathbf{p})$ ,  $\beta = (4\alpha T)^{-1}$ , and the  $S_r^s(x)$  are associated Laguerre polynomials. Truncating the expansion in polynomials by keeping only the leading  $s = 1$  term, the thermal conductivity is given by  $\kappa = 75/16a_{11}$  where

$$a_{11} = \frac{\beta^{9/2}}{2^{5/2} \pi^{1/2}} e^{-\frac{2\omega_0}{T}} \int dv v^7 e^{-\frac{\beta v^2}{2}} \sin^2 \alpha \left( \frac{d\sigma}{d\alpha} \right) d\alpha, \quad (17)$$

in which  $d\sigma/d\alpha$  is the mode-mode scattering cross section. It is expected that the contribution from terms of higher order in  $s$  is small.

We can make a simple estimate of  $\kappa$  by assuming a constant cross section given by  $d\sigma/d\Omega \simeq 1/4k_F^2$ . We find

$$\kappa \simeq \sqrt{\frac{2\alpha T}{\pi}} k_F^2 e^{-\frac{2\omega_0}{T}}, \quad (18)$$

which exposes the temperature scaling of the thermal conductivity when the cross section is independent of  $T$ .

In closing, we note the analogy of the spin diffusive modes with rotons in superfluid helium, which can provide a guide for further studies of scattering and transport involving these collective modes.

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